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# INVARIANT RECORDING OF ELASTICITY THEORY EQUATIONS

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*An invariant (with respect to rotations) formalization of equations of linear and nonlinear elasticity theory is proposed. An equation of state (in the form of a convex generating potential) for various crystallographic systems is written. An algebraic approach is used, which does not require any geometric constructions related to the analysis of symmetry in crystals.*

**Key words:** *elasticity theory, equation of state, crystals, presentation of a group of rotations, Clebsch–Gordan matrix, invariants.*

**Introduction.** We consider the equations of the nonlinear elasticity theory in the following form [1, 2]:

$$\rho\,\frac{\partial u_i}{\partial t} - \frac{\partial s_{ij}}{\partial \xi_j} = 0, \qquad \frac{\partial H_{s_{ij}}}{\partial t} - \frac{\partial u_i}{\partial \xi_j} = 0. \tag{1}$$

Here $s_{ij}$ are the components of the Piola–Kirchhoff tensor [3], which is an asymmetric stress tensor in Lagrangian coordinates $\xi_j$, $u_i$ are the velocities, $\rho = \rho(\xi_1, \xi_2, \xi_3)$ is the density of the medium, $H = H(u_i, s_{ij})$ is the equation of state,

$$H(u_i, s_{ij}) = F_1(u_1, u_2, u_3) + F(s_{11}, s_{12}, \ldots, s_{33}), \tag{2}$$

and $F_1$ and $F$ are certain convex functions. The function $F_1$ is usually taken in the form $F_1 = F_1(u_1, u_2, u_3) = \rho(u_1^2 + u_2^2 + u_3^2)/2$, and the general form of $F$ is determined below (it is shown that $F$ is a function of 11 invariant quadratic forms constructed from quantities transformed in accordance with irreducible presentations of the group of rotations). To describe more complicated processes in an elastic medium, the equation for $H$ can be supplemented by an explicit functional dependence on parameters characterizing the medium:

$$H = H(u_i, s_{ij}, c^{ijkl}). \tag{3}$$

Here $c^{ijkl}$ are the components of the 4-contravariant tensor, which are interpreted as Hooke's constants in the linear elasticity theory. The function $H$ can also depend on the entropy and temperature of the medium.

Introducing the vector of unknowns

$$\boldsymbol{v} = (u_1, u_2, u_3, s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23}, s_{31}, s_{32}, s_{33})^{\mathrm{t}} \tag{4}$$

we can write system (1) as a symmetric hyperbolic system

$$A\frac{\partial \boldsymbol{v}}{\partial t} + \sum_{j=1}^{3} B_j\,\frac{\partial \boldsymbol{v}}{\partial \xi_j} = 0, \tag{5}$$

where $A = A(\xi_1, \xi_2, \xi_3, \boldsymbol{v}) = A^* > 0$ and $B_j = B_j^* = \mathrm{const}$. The matrix $A$ contains the second derivatives of the generating potential (2); positive determinacy of $A$ is a consequence of the convex character of the function $H$. The matrix $A$ of size $12 \times 12$ has the form

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$$A = A^* = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & \tilde{A} & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & \|H_{s_{ij}s_{kl}}\| & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \end{pmatrix}. \tag{6}$$

In addition, the relations

$$\frac{\partial x_i}{\partial t} = u_i, \qquad H_{s_{ij}} = c_{ij} = \frac{\partial x_i}{\partial \xi_j}$$

are valid [$x_i = x_i(t, \xi_1, \xi_2, \xi_3)$ are the Eulerian coordinates and $c_{ij}$ is the distortion matrix], and the law of conservation of energy is satisfied on solutions of system (1):

$$\frac{\partial}{\partial t}\,(u_i H_{u_i} + s_{ij} H_{s_{ij}} - H) = \frac{\partial}{\partial \xi_j}\,(u_i s_{ij}).$$

The objective of the present work is to write the general form of the equations of state (function $H$) for a nonlinear elastic medium, based on the tools of the theory of presentations of the group of rotations. As an example, equations of state for various crystallographic systems are given. It should be noted that the splitting of the vector of unknowns into vectors transformed in accordance with irreducible presentations of the group of rotations is known in the linear theory as the Voigt presentation (see, e.g., [4]) and is used in the theory of crystals. We are not aware, however, of any papers where this presentation in a form similar to that used in the present work is applied to solve nonlinear problems of the elasticity theory. An invariant (with respect to rotations) recording of system (1), consistent with this approach, is also proposed in the paper.

**1. Preliminary Comments.** Let us recall the basic definitions [5] used below. Rotations of a three-dimensional Euclidean space $\mathbb{R}^3$ form the group

$$SO(3) = \{g \in GL(3)\colon \quad g^{\mathrm{t}} g = I_3,\ |g| = 1\},$$

where $GL(3)$ is the group of all non-degenerate matrices of dimension $3 \times 3$ and $I_3$ is the unit matrix. We say that the presentation $T_{SO(3)}$ of the group $SO(3)$ in the space $\mathbb{R}^k$ is defined if each element $g \in SO(3)$ is put into correspondence to a linear transformation $T_g : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$T_{I_3} = I_k, \qquad T_{g_1 \cdot g_2} = T_{g_1} \cdot T_{g_2}.$$

The presentation $T_{SO(3)}$ is called irreducible if $\mathbb{R}^k$ does not contain nontrivial subspaces that are invariant with respect to all transformations $T_g$. The number $N$ (integer or half-integer), such that $k = 2N + 1$, is called the weight of the irreducible presentation. Presentations of integer weights only are used in the present paper.

The Kronecker product $T_g = T_g^1 \times T_g^2$ of presentations $T_g^1$ with a weight $N_1$ and $T_g^2$ with a weight $N_2$ is called the presentation acting on the matrix $B$ of size $(2N_1 + 1) \times (2N_2 + 1)$ by the rule

$$T_g B = T_g^1 B (T_g^2)^{\mathrm{t}}.$$

**Theorem 1.** *If the presentations $T_g^1$ and $T_g^2$ are irreducible, then their product $T_g^1 \times T_g^2$ can be expanded into a direct sum of irreducible presentations with the following weights:*

$$N = |N_1 - N_2|,\ |N_1 - N_2| + 1,\ \ldots,\ N_1 + N_2. \tag{7}$$

This expansion is performed with the use of the Clebsch–Gordan matrices $G^n_{N[N_1,N_2]}$ of size $(2N_1 + 1) \times (2N_2 + 1)$, $n = -N, -N+1, \ldots, N$, which form canonical bases of the corresponding subspaces of the matrix space [6]. These matrices are real, have a rather simple structure (large number of zero elements), are orthonormalized as

$$\mathrm{tr}\,\Big\{(G^n_{N[N_1,N_2]})^{\mathrm{t}} G^m_{N[N_1,N_2]}\Big\} = \delta_{mn},$$

and possess the property of symmetry:

$$G^n_{N[N_1,N_2]} = (-1)^{N+N_1+N_2}(G^n_{N[N_2,N_1]})^{\mathrm{t}} \tag{8}$$

($\delta_{mn}$ is the Kronecker symbol). The matrices are related to each other by recurrent relations, which make it possible to construct the algorithm for calculating these matrices explicitly (this algorithm was realized for arbitrary weights $N_1$ and $N_2$; it was described in detail in [7]).

Let us now consider the Piola–Kirchhoff stress tensor, which is a tensor of the second rank defined in the canonical base by the matrix $T = \|s_{ij}\|_{i,j=1,2,3}$. This tensor can be considered as a Kronecker product of two irreducible presentations of weight 1 of the group of rotations, and it can be expanded into a direct sum of irreducible presentations of weights 0, 1, and 2:

$$T = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = pI_3 + K + S.$$

Here

$$p = (s_{11} + s_{22} + s_{33})/3 = (1/3)\,\mathrm{tr}\,T = \Sigma^{(0)} \tag{9}$$

is the "pressure" (scalar quantity transformed by the presentation of weight 0), $K = -K^*$ (three independent elements of this skew-symmetric matrix form a vector transformed in accordance with an irreducible presentation of weight 1), $S = S^*$, and $\mathrm{tr}\,S = 0$ is the deviator (five independent elements of this symmetric matrix form a vector transformed in accordance with an irreducible presentation of weight 2).

The matrices $K$ and $S$ are expressed via the Clebsch–Gordan matrices as follows:

$$K = -K^* = \begin{bmatrix} 0 & (s_{12}-s_{21})/2 & (s_{13}-s_{31})/2 \\ -(s_{12}-s_{21})/2 & 0 & (s_{23}-s_{32})/2 \\ -(s_{13}-s_{31})/2 & -(s_{23}-s_{32})/2 & 0 \end{bmatrix} = \sum_{j=-1}^{1} \omega_j\, G^j_{1[1,1]}$$

$$= \omega_{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix} + \omega_0 \begin{bmatrix} 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 \end{bmatrix} + \omega_1 \begin{bmatrix} 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We denote the corresponding vector transformed with weight 1 as

$$\mathbf{\Sigma}^{(1)} = \begin{pmatrix} \omega_{-1} \\ \omega_0 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} -(s_{23}-s_{32})/\sqrt{2} \\ (s_{13}-s_{31})/\sqrt{2} \\ -(s_{12}-s_{21})/\sqrt{2} \end{pmatrix}. \tag{10}$$

In Sec. 3, we consider a symmetric stress tensor ($s_{ij} = s_{ji}$). In this case, we obtain $K = -K^* = 0$.

Similarly, the deviator of the stress tensor can be presented as

$$S = \begin{bmatrix} s_{11}-p & (s_{12}+s_{21})/2 & (s_{13}+s_{31})/2 \\ (s_{12}+s_{21})/2 & s_{22}-p & (s_{23}+s_{32})/2 \\ (s_{13}+s_{31})/2 & (s_{23}+s_{32})/2 & s_{33}-p \end{bmatrix} = s_{-2}G^{-2}_{2[1,1]} + s_{-1}G^{-1}_{2[1,1]} + s_0 G^0_{2[1,1]} + s_1 G^1_{2[1,1]} + s_2 G^2_{2[1,1]}.$$

We denote

$$\mathbf{\Sigma}^{(2)} = \begin{pmatrix} s_{-2} \\ s_{-1} \\ s_0 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} -(s_{13}+s_{31})/\sqrt{2} \\ (s_{12}+s_{21})/\sqrt{2} \\ \sqrt{3}\,(s_{22}-p)/\sqrt{2} \\ (s_{23}+s_{32})/\sqrt{2} \\ (s_{11}-s_{33})/\sqrt{2} \end{pmatrix}. \tag{11}$$

This vector is transformed with weight 2. Let us write the corresponding Clebsch–Gordan matrices. Note, these matrices are symmetric with the weights used here ($N_1 = N_2 = 1$ and $N = 2$) and skew-symmetric in the case with $N_1 = N_2 = 1$ and $N = 1$, which corresponds to the property (8):

$$G^{-2}_{2[1,1]} = \begin{bmatrix} 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 \end{bmatrix}, \quad G^{-1}_{2[1,1]} = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G^0_{2[1,1]} = \begin{bmatrix} -1/\sqrt{6} & 0 & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & -1/\sqrt{6} \end{bmatrix},$$

$$G^1_{2[1,1]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{bmatrix}, \quad G^2_{2[1,1]} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} \end{bmatrix}.$$

In what follows, we use the vectors of the unknowns (9)–(11) [or (9) and (11) for $s_{ij} = s_{ji}$ in the case of a symmetric stress tensor $T$].

We introduce new notation for velocities:

$$\boldsymbol{v}^{(1)} = \begin{pmatrix} u_{-1} \\ u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{12}$$

**2. Structure of the Equation of State.** Let us set the general form of the function $F(s_{11}, s_{12}, \ldots, s_{33})$ involved into the equation of state (2). The generating potential $H$ has to be independent of the coordinate system; hence, the function $F$ is a function of invariants. To write the full set of invariants depending on the variables $\{s_{ij}\}$, we find which invariant quadratic forms (and how many forms) can be composed of the variables (9)–(11). For this purpose, we write all possible products of weights 0, 1, and 2 ($0 \times 0$, $0 \times 2$, $2 \times 0$, $2 \times 2$, $0 \times 1$, $1 \times 0$, $1 \times 1$, $1 \times 2$, and $2 \times 1$) and their expansions into irreducible presentations. It should be noted that the expansions for the cases $0 \times 2$ and $2 \times 0$, $0 \times 1$ and $1 \times 0$, and $1 \times 2$ and $2 \times 1$ coincide by virtue of the commutative character of the Kronecker product. According to Eq. (7), in the case of expansion of the product of weights $N_1$ and $N_2$ into irreducible presentations, there arise

$$(2|N_1 - N_2| + 1) + \ldots + [2(N_1 + N_2) + 1] = (2N_1 + 1)(2N_2 + 1) \tag{13}$$

parameters $\boldsymbol{w}^{(N)}_n$ transformed in accordance with irreducible presentations with the corresponding weights $N$.

Let us write the formula of expansion of the Kronecker product of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ transformed in accordance with irreducible presentations with the weights $N_1$ and $N_2$ into irreducible presentations [8]:

$$\boldsymbol{p}^{(N_1)} \times \boldsymbol{q}^{(N_2)} = \sum_{N=|N_1-N_2|}^{N_1+N_2} \Big( \sum_{n=-N}^{N} \boldsymbol{w}^{(N)}_n G^n_{N[N_1,N_2]} \Big).$$

The invariant quadratic forms (transformed in accordance with the presentation of the zero weight) composed of these vectors are written in the following manner (the invariance follows from the orthogonality of the presentations considered):

$$\begin{gathered} I_{(N)} = \sum_{n=-N}^{N} \boldsymbol{w}^{(N)}_n ([\boldsymbol{p}^{(N_1)}]^{\mathrm{t}} G^n_{N[N_1,N_2]}, \boldsymbol{q}^{(N_2)}), \\ N = |N_1 - N_2|,\ |N_1 - N_2| + 1,\ \ldots,\ N_1 + N_2. \end{gathered} \tag{14}$$

Thus, we can determine the number of parameters of the elastic medium and write all possible invariant quadratic forms (through these forms, we can express the energy of the elastic medium, i.e., the generating potential $H$). In the case of the linear elasticity theory, the parameters $\boldsymbol{w}^{(N)}_n$ (corresponding to weights 0 and 2) can be interpreted as constants in Hooke's law for an elastic solid possessing some types of symmetry (see Sec. 3).

Thus, according to Eq. (13), we have:

1) $0 \times 0 \Longrightarrow 1$ parameter;
2) $0 \times 2 \Longrightarrow 5$ parameters;
3) $2 \times 0 \Longrightarrow 5$ parameters;
4) $2 \times 2 \Longrightarrow 1 + 3 + 5 + 7 + 9 = 25$ parameters.

Hence, there are 36 parameters for media with a symmetric stress tensor $T = T^*$.

In the general case, there is also weight 1 and 45 more parameters:

5) $0 \times 1 \Longrightarrow 3$ parameters;

6) $1 \times 0 \Longrightarrow 3$ parameters;
7) $1 \times 1 \Longrightarrow 1 + 3 + 5$ parameters;
8) $1 \times 2 \Longrightarrow 3 + 5 + 7$ parameters;
9) $2 \times 1 \Longrightarrow 3 + 5 + 7$ parameters.
Hence, there are $36 + 45 = 81$ parameters.

We write the corresponding quadratic forms, using Eq. (14). [Note, of interest for the present work are only different quadratic forms with symmetric matrices forming the linear part of the matrix $\tilde{A} = \tilde{A}^*$ from (6).] Therefore, there are only 45 constants left out of 81 (in the case of a symmetric stress tensor, 21 out of 36).

1. For $N_1 = 0$, $N_2 = 0$, and $N = 0$, we have $G^0_{0[0,0]} = [1]$, and the corresponding quadratic form is

$$I_{(1)} = c_1 G^0_{0[0,0]} p^2.$$

2. For $N_1 = 0$, $N_2 = 2$, and $N = 2$, we have $G^0_{2[0,2]} = [0\ 0\ 1\ 0\ 0]$, $G^{-1}_{2[0,2]} = [0\ 1\ 0\ 0\ 0]$, $G^1_{2[0,2]} = [0\ 0\ 0\ 1\ 0]$, $G^{-2}_{2[0,2]} = [1\ 0\ 0\ 0\ 0]$, and $G^2_{2[0,2]} = [0\ 0\ 0\ 0\ 1]$, and the quadratic form is

$$I_{(2)} = \sum_{j=-2}^{2} a_j G^j_{2[0,2]} \Sigma^{(2)} p = \sum_{j=-2}^{2} a_j \Sigma^{(2)}_j p$$

($a_j$ are five arbitrary parameters).

3. For $N_1 = 2$, $N_2 = 0$, and $N = 2$, the quadratic form is the same as that in case 2:

$$I_{(2)} = \sum_{j=-2}^{2} a_j (G^j_{2[2,0]} p, \Sigma^{(2)}).$$

4. For $N_1 = 2$ and $N_2 = 2$, there are several options.

4.1. For $N = 0$,

$$I_{(3)} = c_2 (G^0_{0[2,2]} \Sigma^{(2)}, \Sigma^{(2)}) = \tilde{c}_2 (\Sigma^{(2)}, \Sigma^{(2)})$$

($G_{0[2,2]}$ is a diagonal matrix).

4.2. For $N = 1$, the quadratic form is $\sum\limits_{j=-1}^{1} \tilde{a}_j (G^j_{1[2,2]} \Sigma^{(2)}, \Sigma^{(2)})$, where

$$G^{-1}_{1[2,2]} = \begin{bmatrix} 0 & -1/\sqrt{10} & 0 & 0 & 0 \\ 1/\sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}/\sqrt{10} & 0 \\ 0 & 0 & \sqrt{3}/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 0 & 0 & -1/\sqrt{10} & 0 \end{bmatrix},$$

$$G^{1}_{1[2,2]} = \begin{bmatrix} 0 & 0 & 0 & 1/\sqrt{10} & 0 \\ 0 & 0 & -\sqrt{3}/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & \sqrt{3}/\sqrt{10} & 0 & 0 & 0 \\ -1/\sqrt{10} & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{10} & 0 & 0 & 0 \end{bmatrix},$$

$$G^{0}_{1[2,2]} = \begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{2}/\sqrt{5} \\ 0 & 0 & 0 & 1/\sqrt{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{10} & 0 & 0 & 0 \\ -\sqrt{2}/\sqrt{5} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note, by virtue of property (8), these matrices are skew-symmetric, and this case is not considered further.

4.3. For $N = 2$, we have

$$I_{(4)} = \sum_{j=-2}^{2} b_j (G^j_{2[2,2]} \Sigma^{(2)}, \Sigma^{(2)}),$$

where

$$G^{-2}_{2[2,2]} = \begin{bmatrix} 0 & 0 & -\sqrt{2}/\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 \\ -\sqrt{2}/\sqrt{7} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G^{2}_{2[2,2]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/\sqrt{7} \\ 0 & 0 & 0 & -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 \\ 0 & 0 & -\sqrt{2}/\sqrt{7} & 0 & 0 \end{bmatrix},$$

$$G^{-1}_{2[2,2]} = \begin{bmatrix} 0 & 0 & 0 & -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 \\ 0 & 0 & 1/(\sqrt{2}\sqrt{7}\,) & 0 & \sqrt{3}/(\sqrt{2}\sqrt{7}\,) \\ 0 & 1/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 \\ -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 \end{bmatrix},$$

$$G^{1}_{2[2,2]} = \begin{bmatrix} 0 & -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 \\ -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(\sqrt{2}\sqrt{7}\,) & 0 \\ 0 & 0 & 1/(\sqrt{2}\sqrt{7}\,) & 0 & -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) \\ 0 & 0 & 0 & -\sqrt{3}/(\sqrt{2}\sqrt{7}\,) & 0 \end{bmatrix},$$

$$G^{0}_{2[2,2]} = \begin{bmatrix} -\sqrt{2}/\sqrt{7} & 0 & 0 & 0 & 0 \\ 0 & 1/(\sqrt{2}\sqrt{7}\,) & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}/\sqrt{7} & 0 & 0 \\ 0 & 0 & 0 & 1/(\sqrt{2}\sqrt{7}\,) & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/\sqrt{7} \end{bmatrix}.$$

4.4. For $N = 3$, we obtain the quadratic form $\sum\limits_{j=-3}^{3} \tilde{b}_j (G^j_{3[2,2]}\Sigma^{(2)}, \Sigma^{(2)})$. The corresponding matrices are also skew-symmetric (in what follows, we do not write them in explicit form).

4.5. For $N = 4$, we have

$$I_{(5)} = \sum_{j=-4}^{4} d_j (G^j_{4[2,2]}\Sigma^{(2)}, \Sigma^{(2)}).$$

Thus, to enumerate all paired different symmetric quadratic forms composed by the above-indicated method for a symmetric stress tensor (with weights 0 and 2 only), we need to eliminate cases 3, 4.2, and 4.4 from the above-given cases; as a result, there remain $36 - 5 - 3 - 7 = 21$ parameters. The remaining cases 1, 2, 4.1, 4.3, and 4.5 correspond to five invariants $I_{(1)}, I_{(2)}, \ldots, I_{(5)}$, which may affect the energy.

Let us continue considerations for the general case by adding weight 1.

5. For $N_1 = 0$, $N_2 = 1$, and $N = 1$, we have

$$I_{(6)} = \sum_{j=-1}^{1} \alpha_j G^j_{1[0,1]}\Sigma^{(1)} p.$$

6. For $N_1 = 1$, $N_2 = 0$, and $N = 1$, the invariant quadratic form is the same as that in case 5, because $G^j_{1[0,1]} = (G^j_{1[1,0]})^{\mathrm{t}}$. Therefore, case 6 is not taken into account, and the number of parameters decreases by 3.

7. For $N_1 = N_2 = 1$, several options are possible.

7.1. For $N = 0$, we have

$$I_{(7)} = c_3(G^0_{0[1,1]}\Sigma^{(1)}, \Sigma^{(1)}) = \tilde{c}_3(\Sigma^{(1)}, \Sigma^{(1)}).$$

7.2. For $N = 1$, we obtain the quadratic form $\sum\limits_{j=-1}^{1} \tilde{\beta}_j (G^j_{1[1,1]}\Sigma^{(1)}, \Sigma^{(1)})$, and the matrices $G^j_{1[1,1]}$ are skew-symmetric.

7.3. For $N = 2$, we have

$$I_{(8)} = \sum_{j=-2}^{2} \beta_j (G^j_{2[1,1]}\Sigma^{(1)}, \Sigma^{(1)}).$$

8. For $N_1 = 2$ and $N_2 = 1$, several options are possible.

8.1. For $N = 1$, we have

$$I_{(9)} = \sum_{j=-1}^{1} f_j (G^j_{1[2,1]}\Sigma^{(1)}, \Sigma^{(2)}).$$

8.2. For $N = 2$, we have

$$I_{(10)} = \sum_{j=-2}^{2} g_j (G^j_{2[2,1]}\Sigma^{(1)}, \Sigma^{(2)}).$$

8.3. For $N = 3$, we have

$$I_{(11)} = \sum_{j=-3}^{3} h_j (G^j_{3[2,1]}\Sigma^{(1)}, \Sigma^{(2)}).$$

9. For $N_1 = 1$ and $N_2 = 2$, the quadratic forms coincide with the quadratic forms in case 8, i.e., the corresponding $3 + 5 + 7 = 15$ constants are ignored.

Thus, ignoring the "extra" cases 6, 7.1, and 9, we obtain $45 - 3 - 3 - 15 = 24$ additional parameters. With allowance for cases 1–4, we have $21 + 24 = 45$ independent parameters, which may affect the generating potential (2). There are six more invariants if we take into account cases 5, 7.1, 7.3, and 8.1–8.3. Thus, the sought function $F$ in the expression for the energy of the nonlinear elastic medium can depend on 11 invariants determined with accuracy to an arbitrary constant factor (with allowance for this fact, we admit some arbitrariness in the notation of parameters). The equation of state has the form

$$H = \rho(u^2_{-1} + u^2_0 + u^2_1)/2 + F(I_{(1)}, I_{(2)}, I_{(3)}, \ldots, I_{(11)}). \tag{15}$$

The parameters

$$\begin{gathered} c_1,\ c_2,\ a_j\ (j = -2, \ldots, 2),\ b_j\ (j = -2, \ldots, 2),\ d_j\ (j = -4, \ldots, 4), \\ c_3,\ \beta_j\ (j = -2, \ldots, 2),\ \alpha_j\ (j = -1, \ldots, 1),\ f_j\ (j = -1, \ldots, 1), \\ g_j\ (j = -2, \ldots, 2),\ h_j\ (j = -3, \ldots, 3) \end{gathered} \tag{16}$$

involved into invariants $I_{(1)}$–$I_{(11)}$ characterize the medium and can depend on spatial variables.

Note, for an isotropic case, there remain only three invariants corresponding to the zero weight:

$$I_{(1)} = c_1 p^2, \qquad I_{(3)} = c_2(\Sigma^{(2)}, \Sigma^{(2)}), \qquad I_{(7)} = c_3(\Sigma^{(1)}, \Sigma^{(1)}). \tag{17}$$

For Eqs. (1) to be well-posed, the function $H$ has to be convex over the initial variables (4).

The necessary and sufficient conditions of convexity of the function $H$ [imposed onto parameters (16)] were studied in [1] for a function of a somewhat different type. References to papers dealing with the positive determinacy of the energy matrix for the linear case in standard variables can be found, e.g., in [9].

As predicted by Eq. (3), let the function $H$ explicitly depend on the medium parameters $c^{ijkl}$; the set of these parameters is divided into groups in accordance with Eq. (16). Using this splitting and Eqs. (14) and (8), we can construct invariant quadratic forms from the parameters $c^{ijkl}$, which may affect the convex generating potential $H$, by analogy with construction of the quadratic forms $I_{(1)}$–$I_{(11)}$. For instance, three invariants can be composed from the vectors $\boldsymbol{a} = (a_{-2}, a_{-1}, \ldots, a_2)$ and $\boldsymbol{b} = (b_{-2}, b_{-1}, \ldots, b_2)$:

$$\sum_{n=-N}^{N} \varphi_N^n (G_{N[2,2]}^n \boldsymbol{a}, \boldsymbol{b}), \qquad N = 0, 2, 4$$

($\varphi_N^n$ are some additional parameters characterizing the medium).

**3. Equation of State for a Crystalline Medium.** Let us consider a linear elastic medium with a symmetric stress tensor whose components are re-denoted by $T = \|\sigma_{ij}\| = \|\sigma_{ji}\| = T^*$. The vectors of the unknowns (9) and (11), which are transformed with weights 0 and 2, respectively, are written as follows:

$$\boldsymbol{u}^{(0)} = p = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}, \qquad \boldsymbol{u}^{(2)} = \begin{pmatrix} \sigma_{-2} \\ \sigma_{-1} \\ \sigma_0 \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\,\sigma_{13} \\ \sqrt{2}\,\sigma_{12} \\ \sqrt{6}\,(\sigma_{22} - p)/2 \\ \sqrt{2}\,\sigma_{23} \\ (\sigma_{11} - \sigma_{33})/\sqrt{2} \end{pmatrix}. \tag{18}$$

Now we do not have the vector transformed with weight 1 and corresponding to the skew-symmetric part of $T$. The expression for the internal energy of the elastic solid can now be written as

$$E = (1/2)\varepsilon_{ij}\sigma_{ij} = c^{ijkl}\sigma_{kl}\sigma_{ij},$$

where $\varepsilon_{ij}$ is the tensor of small strains and $c^{ijkl}$ is the tensor of rank 4 (Hooke's parameters) consisting in the general case of 81 constants. By virtue of the symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$, we obtain 36 constants; in addition, there is also the symmetry $c^{ijkl} = c^{klij}$; finally, we obtain 21 different constants. The same result is obtained with the use of irreducible presentations of the group of rotations.

In addition, it was established in Sec. 2 that

$$E = E(I_{(1)}, I_{(2)}, \ldots, I_{(5)}) = I_{(1)} + I_{(2)} + \ldots + I_{(5)}$$

(in this case, as the energy is a quadratic form, it is a linear combination of invariants; as the invariants are determined with accuracy to an arbitrary constant factor, the expression for the energy can be written as a sum of these invariants). Let us write the matrices of the quadratic forms composing the invariants $I_{(1)}$–$I_{(5)}$ by passing from variables (18) to the "classical" variables:

$$\boldsymbol{u} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{13}, \sigma_{12}, \sigma_{23})^{\mathrm{t}}.$$

Let us recall that there are two invariants of the power, equal to or smaller than two, with respect to $\sigma_{ij}$ for an isotropic elastic solid (i.e., if the solid is invariant with respect to all rotations of the space):

$$J_{(1)} = \sigma_{ii}, \qquad J_{(2)} = \sigma_{ij}\sigma_{ij}.$$

From these invariants, we can compose a quadratic form containing two different parameters of the medium (out of possible 21 parameters) of the form

$$E_0 = c_1(\lambda, \mu)J_{(1)}^2 + c_2(\lambda, \mu)J_{(2)} = (A_0\boldsymbol{u}, \boldsymbol{u})$$

($\lambda$ and $\mu$ are the Lamé constants). The same quadratic form can be obtained from cases 1 and 4.1 for the zero weight ($N = 0$) considered in Sec. 2:

$$E_0 = \tilde{c}_1 G_{0[0,0]}^0 p^2 + \tilde{c}_2 (G_{0[2,2]}^0 \boldsymbol{u}^{(2)}, \boldsymbol{u}^{(2)}) = I_{(1)} + I_{(2)} = (C_0\boldsymbol{u}, \boldsymbol{u}).$$

Here

$$C_0 = C_0^* = \begin{bmatrix} c_1 + c_2 & c_1 & c_1 & 0 & 0 & 0 \\ & c_1 + c_2 & c_1 & 0 & 0 & 0 \\ & & c_1 + c_2 & 0 & 0 & 0 \\ & & & c_2 & 0 & 0 \\ & \text{sym} & & & c_2 & 0 \\ & & & & & c_2 \end{bmatrix}, \tag{19}$$

and the term $E_0$ always enters the expression for the energy $E$.

We write the quadratic form for case 2:

$$I_{(3)} = \sum_{j=-2}^{2} a_j G^j_{2[0,2]} \boldsymbol{u}^{(2)} p = (C_1 \boldsymbol{u}, \boldsymbol{u}).$$

Here

$$C_1 = C_1^* = \begin{bmatrix} -a_0/(2\sqrt{3}\,) + a_2/2 & a_0/(2\sqrt{3}\,) + a_2/2 & -a_0/\sqrt{3} & -a_{-2} & a_{-1} & a_1 \\ & a_0/\sqrt{3} & a_0/(2\sqrt{3}\,) - a_2/2 & -a_{-2} & a_{-1} & a_1 \\ & & -a_0/(2\sqrt{3}\,) - a_2/2 & -a_{-2} & a_{-1} & a_1 \\ & & & 0 & 0 & 0 \\ & \text{sym} & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}.$$

For case 4.3, we have

$$I_{(4)} = \sum_{j=-2}^{2} b_j (G^j_{2[2,2]} \boldsymbol{u}^{(2)}, \boldsymbol{u}^{(2)}) = (C_2 \boldsymbol{u}, \boldsymbol{u}),$$

where

$$C_2 = C_2^* = \begin{bmatrix} -\dfrac{b_0}{3} + \dfrac{b_2}{\sqrt{3}} & -\dfrac{2b_0}{3} - \dfrac{2b_2}{\sqrt{3}} & \dfrac{b_0}{3} & -\dfrac{2b_{-2}}{\sqrt{3}} & \dfrac{2b_{-1}}{\sqrt{3}} & -\dfrac{4b_1}{\sqrt{3}} \\ & \dfrac{2b_0}{3} & -\dfrac{2b_0}{3} + \dfrac{2b_2}{\sqrt{3}} & \dfrac{4b_{-2}}{\sqrt{3}} & \dfrac{2b_{-1}}{\sqrt{3}} & \dfrac{2b_1}{\sqrt{3}} \\ & & -\dfrac{b_0}{3} - \dfrac{b_2}{\sqrt{3}} & -\dfrac{2b_{-2}}{\sqrt{3}} & -\dfrac{4b_{-1}}{\sqrt{3}} & \dfrac{2b_1}{\sqrt{3}} \\ & & & -2b_0 & 2\sqrt{3}\,b_1 & 2\sqrt{3}\,b_{-1} \\ & \text{sym} & & & b_0 + \sqrt{3}\,b_2 & -2\sqrt{3}\,b_{-2} \\ & & & & & b_0 - \sqrt{3}\,b_2 \end{bmatrix}$$

(the matrix also contains five independent parameters chosen with accuracy to a common constant factor).

For case 4.5, we have

$$I_{(5)} = \sum_{j=-4}^{4} d_j (G^j_{4[2,2]} \boldsymbol{u}^{(2)}, \boldsymbol{u}^{(2)}) = (C_3 \boldsymbol{u}, \boldsymbol{u}).$$

We will not write the matrix $C_3$ because it is too cumbersome. It contains nine elastic constants $d_j$ $(j = -4, \dots, 4)$.

If there are no additional rotational symmetries (except for the identical transformation) for the elastic solid, the expression for the energy includes all four terms given above,

$$E = E_0 + I_{(3)} + I_{(4)} + I_{(5)} = (C\boldsymbol{u}, \boldsymbol{u}) \tag{20}$$

and contains 21 independent parameters. Here $C = C_0 + C_1 + C_2 + C_3$.

Thus, the parameters obtained (a total of 21 parameters) can be divided into groups in the following manner:

1) $c_1$ and $c_2$ are two quantities invariant with respect to all rotations and transformed with weight 0;

2) $\boldsymbol{a} = (a_{-2}, a_{-1}, a_0, a_1, a_2)$ is a five-dimensional vector transformed with weight 2;

3) $\boldsymbol{b} = (b_{-2}, b_{-1}, b_0, b_1, b_2)$ is a five-dimensional vector transformed with weight 2;

4) $\boldsymbol{d} = (d_{-4}, d_{-3}, d_{-2}, d_{-1}, d_0, d_1, d_2, d_3, d_4)$ is a nine-dimensional vector transformed with weight 4.

These parameters are contained in the matrix $C$ of the quadratic form of energy (20). Let us indicate the relations between these parameters in the case of symmetry in crystals. For this purpose, we use the available results for seven crystallographic systems (syngonies) given, for instance, in [10]. Equating to zero those elements of the matrix $C$ that have zero values according to these results, we obtain an individual (possibly, overdetermined) system of linear equations for each case; the solution of the system is expected to yield the sought relations. Let us introduce the notations for the transformations responsible for rotation by $360^\circ/n$ around the $x_3$ axis ($R_n$), half-rotation (i.e., rotation by $180^\circ$) around the $x_2$ axis ($L_2$), half-rotation around the bisector $x_1 = x_2$, $x_3 = 0$ ($L$), and rotation around the diagonal of the cube $x_1 = x_2 = x_3$ ($S$).

Let us consider each of seven crystallographic systems.

1. Triclinic system $\{I\}$. The corresponding group of transformations consists of the identical transformation only, i.e., there are 21 independent parameters, i.e., $E = (C\boldsymbol{u}, \boldsymbol{u})$, where $C$ is the full matrix.

2. Monoclinic system $\langle R_2 \rangle$ generated by rotation by 180° around one axis. For this system, we obtain a system of eight linear equations (eight unknowns), which yields

$$a_{-2} = b_{-2} = d_{-4} = d_{-2} = a_1 = b_1 = d_1 = d_3 = 0.$$

Thus, we have $21 - 8 = 13$ independent parameters located in the above-described vectors as follows:

$$c_1, c_2, (0, a_{-1}, a_0, 0, a_2), (0, b_{-1}, b_0, 0, b_2), (0, d_{-3}, 0, d_{-1}, d_0, 0, d_2, 0, d_4).$$

3. Rhombic system $\langle R_2, L_2 \rangle$. To relations valid in case 2, we need to add the relations $a_{-1} = b_{-2} = d_{-1} = d_{-3} = 0$, i.e., we have $13 - 4 = 9$ independent parameters:

$$c_1, c_2, (0, 0, a_0, 0, a_2), (0, 0, b_0, 0, b_2), (0, 0, 0, 0, d_0, 0, d_2, 0, d_4).$$

4. Trigonal system $\langle R_3 \rangle$, $\langle R_3, L_2 \rangle$. In this case, we have 14 equations with 19 unknowns. Solving this overdetermined system, we obtain $21 - 14 = 7$ independent parameters $c_1$, $c_2$, $a_0$, $b_{-2}$, $b_0$, $b_1$, and $d_0$:

$$c_1, c_2, (0, 0, a_0, 0, \sqrt{3}\, a_0), (b_{-2}, 0, b_0, b_1, \sqrt{3}\, b_0),$$

$$(-(\sqrt{7}/\sqrt{3}\,)b_{-2}, 0, -(5/\sqrt{3}\,)b_{-2}, 0, d_0, \sqrt{2}\sqrt{3}\, b_1, (2\sqrt{5}/3)d_0, (\sqrt{7}\sqrt{2}/\sqrt{3})b_1, (\sqrt{5}\sqrt{7}/3)d_0).$$

5. Tetragonal system $\langle R_4 \rangle$, $\langle R_4, L_2 \rangle$. In this case, we also obtain seven independent parameters $c_1$, $c_2$, $a_0$, $b_0$, $d_{-1}$, $d_0$, and $d_2$ located as follows:

$$c_1, c_2, (0, 0, a_0, 0, \sqrt{3}\, a_0), (0, 0, b_0, 0, \sqrt{3}\, b_0),$$

$$(0, (1/\sqrt{7})d_{-1}, 0, d_{-1}, d_0, 0, d_2, 0, -(\sqrt{5}/\sqrt{7}\,)d_0 - (2/\sqrt{7}\,)d_2).$$

6. Hexagonal system $\langle R_6 \rangle$, $\langle R_6, L_2 \rangle$. In this case, we have five independent parameters $c_1$, $c_2$, $a_0$, $b_0$, and $d_0$ located as follows:

$$c_1, c_2, (0, 0, a_0, 0, \sqrt{3}\, a_0), (0, 0, b_0, 0, \sqrt{3}\, b_0), (0, 0, 0, 0, d_0, 0, (2\sqrt{5}/3)d_0, 0, -(\sqrt{5}\sqrt{7}/3)d_0).$$

7. Cubic system $\langle S, R_2 \rangle$, $\langle S, L \rangle$. In this case, we obtain $\boldsymbol{a} = \boldsymbol{b} = 0$ and three independent parameters located as follows:

$$c_1, c_2, \boldsymbol{d} = (0, 0, 0, 0, d_0, 0, 0, 0, -(\sqrt{5}/\sqrt{7})d_0).$$

Thus, for each crystallographic system, we can write a quadratic form for the equation of state in explicit form in terms of irreducible presentations of the group of rotations. Note that the approach proposed for constructing the equation of state is purely algebraic, in contract to the commonly used geometric approach. The geometric interpretation of relations for Hooke's parameters described in this section (for instance, their relation with parameters defining the corresponding crystalline lattice [11]) is an independent problem.

**4. Invariant Recording of the Equations.** In this section, we use the statements [5, 6, 8] about the general form of the system of the first-order partial differential equations, which is invariant with respect to rotations.

Let us consider the equations of the linear elasticity theory in the form

$$\rho\, \frac{\partial u_i}{\partial t} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0,$$

$$\frac{\partial \varepsilon_{ij}}{\partial t} - \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) = 0,$$

where $\varepsilon_{ij} = \varepsilon_{ji}$ and $\sigma_{ij} = \sigma_{ji}$ ($i = 1, 2, 3$; $j = 1, 2, 3$). From Hooke's law $\varepsilon_{ij} = c^{ijkl}\sigma_{kl}$, we obtain

$$\begin{gathered} \rho\, \frac{\partial u_i}{\partial t} - \frac{\partial \sigma_{ij}}{\partial x_i} = 0, \\ c^{ijkl}\, \frac{\partial \sigma_{kl}}{\partial t} - \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) = 0. \end{gathered} \tag{21}$$

By introducing the vector of unknowns $\tilde{\boldsymbol{v}} = (u_1, u_2, u_3, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})$, we can write these equations in the form of a symmetric hyperbolic system similar to (5), where the matrix $A = A(x_1, x_2, x_3) = A^* > 0$ of size $9 \times 9$ depends on the medium parameters, which can have different values at each point of the medium.

In the isotropic case (if the medium is invariant with respect to all rotations of the coordinate system), the matrix $\tilde{A}$ of size $6 \times 6$ from (6) has the form (19). The matrix $\tilde{A}$ contains two parameters ($c_1$ and $c_2$) characterizing the medium properties and expressed via the Lamé parameters $\lambda$ and $\mu$: $c_1 = -\lambda/[2\mu(3\lambda + 2\mu)]$ and $c_2 = 1/\mu$. In the anisotropic case, there appear additional parameters. Thus, if there are no additional symmetries in the medium, there are 21 independent parameters (the maximum number), and the matrix $\tilde{A}$ coincides with the filled matrix $C$ in (20). Moreover, certain intermediate cases are possible (various crystallographic systems considered in Sec. 3).

In variables (12) and (18), Eqs. (21) for an isotropic medium are written as

$$\begin{gathered} A_1 \frac{\partial}{\partial t} \boldsymbol{v}^{(1)} + \Delta_- \boldsymbol{u}^{(2)} + \Delta_+ \boldsymbol{u}^{(0)} = 0, \\ A_0 \frac{\partial}{\partial t} \boldsymbol{u}^{(0)} + \Delta_- \boldsymbol{v}^{(1)} = 0, \\ A_2 \frac{\partial}{\partial t} \boldsymbol{u}^{(2)} + \Delta_+ \boldsymbol{v}^{(1)} = 0. \end{gathered} \tag{22}$$

Here

$$\Delta_- \boldsymbol{u}^{(L)} = c_-(L) \sum_{i=-1}^{1} \frac{\partial}{\partial x_i} G^i_{1[L-1,L]} \boldsymbol{u}^{(L)}, \qquad \Delta_+ \boldsymbol{u}^{(L)} = c_+(L) \sum_{i=-1}^{1} \frac{\partial}{\partial x_i} G^i_{1[L+1,L]} \boldsymbol{u}^{(L)}$$

are the matrix differential operators containing the Clebsch–Gordan matrices, which are invariant with respect to rotations. The first operator decreases the weight of the vector transformed in accordance with the irreducible presentation of the weight $L$ of the group of rotations by unity (it is an analog of the div operator), and the second operator increases the weight of this vector by unity (it is an analog of the grad operator):

$$c_-(1) = c_+(0) = -1, \quad c_+(1) = \frac{\sqrt{2}\sqrt{5}}{\sqrt{3}}, \quad c_-(2) = \frac{2\sqrt{5}}{\sqrt{3}} = \sqrt{2} c_+(1), \quad A_1 = \begin{pmatrix} \rho & & \\ & \rho & \\ & & \rho \end{pmatrix}.$$

For an isotropic medium, we have

$$A_0 = \hat{c}_1 G^0_{0[0,0]} = \hat{c}_1 = \frac{1}{3\lambda + 2\mu}, \quad A_2 = \hat{c}_2 G^0_{0[2,2]} = \begin{pmatrix} 1/\mu & & & & \\ & 1/\mu & & & \\ & & 1/\mu & & \\ & & & 1/\mu & \\ & & & & 1/\mu \end{pmatrix}.$$

In the case of an anisotropic medium, system (22) is slightly more complicated: the first subsystem (for weight 1) remains unchanged, and two other subsystems are united:

$$\hat{A} \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} + \begin{pmatrix} \Delta_- \boldsymbol{v}^{(1)} & 0 \\ 0 & \Delta_+ \boldsymbol{v}^{(1)} \end{pmatrix} = 0. \tag{23}$$

Here $\tilde{\boldsymbol{u}} = (\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(2)})^{\mathrm{t}}$ is the vector consisting of six components (18),

$$\hat{A} = \begin{pmatrix} A_0 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} 0 & \sum\limits_{j=-2}^{2} a_j G^j_{2[0,2]} \\ \sum\limits_{j=-2}^{2} a_j G^j_{2[2,0]} & \sum\limits_{j=-2}^{2} b_j G^j_{2[2,2]} + \sum\limits_{j=-4}^{4} d_j G^j_{4[2,2]} \end{pmatrix}.$$

Hooke's parameters involved into the matrix $\hat{A}$ at the derivative $\partial/\partial t$ are divided into groups. If the crystals possess certain kinds of symmetry, these parameters are related by additional formulas given in Sec. 3. As an example, let us write the matrix $\hat{A}$ for the case with cubic symmetry. In such variables, the matrix $\hat{A}$ is a diagonal matrix, which facilitates calculation of the characteristics of the equations considered:

$$\hat{A} = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}}\,c_2 - \frac{6\sqrt{2}}{\sqrt{5}\sqrt{7}}\,d_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}}\,c_2 - \frac{6\sqrt{2}}{\sqrt{5}\sqrt{7}}\,d_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{5}}\,c_2 + \frac{9\sqrt{2}}{\sqrt{5}\sqrt{7}}\,d_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}}\,c_2 - \frac{6\sqrt{2}}{\sqrt{5}\sqrt{7}}\,d_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}}\,c_2 + \frac{9\sqrt{2}}{\sqrt{5}\sqrt{7}}\,d_0 \end{bmatrix}.$$

Generalizing the invariant recording proposed here to the nonlinear case, we can assume that system (1) in variables (9)–(12) is written as

$$\begin{gathered} \rho\,\frac{\partial}{\partial t}\,\boldsymbol{v}^{(1)} + \Delta_{-}\boldsymbol{\Sigma}^{(2)} + \Delta_{+}\boldsymbol{\Sigma}^{(0)} + \Delta_0\boldsymbol{\Sigma}^{(1)} = 0, \\ \hat{A}\,\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{\Sigma}^{(0)} \\ \boldsymbol{\Sigma}^{(1)} \\ \boldsymbol{\Sigma}^{(2)} \end{pmatrix} + \begin{pmatrix} \Delta_{-}\boldsymbol{v}^{(1)} & 0 & 0 \\ 0 & \Delta_0\boldsymbol{v}^{(1)} & 0 \\ 0 & 0 & \Delta_{+}\boldsymbol{v}^{(1)} \end{pmatrix} = 0, \end{gathered} \tag{24}$$

where we have the following relations for each vector $\boldsymbol{u}^{(L)}$ transformed in accordance with the weight $L$:

$$\Delta_{-}\boldsymbol{u}^{(L)} = c_{-}(L)\sum_{i=-1}^{1}\frac{\partial}{\partial\xi_i}\,G^i_{1[L-1,L]}\boldsymbol{u}^{(L)}, \qquad \Delta_{+}\boldsymbol{u}^{(L)} = c_{+}(L)\sum_{i=-1}^{1}\frac{\partial}{\partial\xi_i}\,G^i_{1[L+1,L]}\boldsymbol{u}^{(L)},$$

$$\Delta_0\boldsymbol{u}^{(L)} = c_0(L)\sum_{i=-1}^{1}\frac{\partial}{\partial\xi_i}\,G^i_{1[L,L]}\boldsymbol{u}^{(L)}.$$

The latter operator retains the weight of the vector and is a multidimensional analog of the rot operator. This operator appears in the system recording owing to addition of the skew-symmetric part of the stress tensor. The matrix at the derivative $\partial/\partial t$ (of size $9\times 9$) has the form

$$\hat{A} = \begin{pmatrix} H_{pp} & \|H_{p\Sigma^{(1)}_i}\|_{i=-1,0,1} & \|H_{p\Sigma^{(2)}_k}\|_{k=-2,\dots,2} \\ \|H_{\Sigma^{(1)}_i p}\|_{i=-1,0,1} & \|H_{\Sigma^{(1)}_i\Sigma^{(1)}_j}\|_{i,j=-1,0,1} & \|H_{\Sigma^{(1)}_i\Sigma^{(2)}_k}\|_{i=-1,0,1,k=-2,\dots,2} \\ \|H_{\Sigma^{(2)}_k p}\|_{k=-2,\dots,2} & \|H_{\Sigma^{(2)}_k\Sigma^{(1)}_i}\|_{i=-1,0,1,k=-2,\dots,2} & \|H_{\Sigma^{(2)}_k\Sigma^{(2)}_l}\|_{k,l=-2,\dots,2} \end{pmatrix},$$

$$\hat{A} = \hat{A}(\xi_{-1},\xi_0,\xi_1,\boldsymbol{v}^{(1)},p,\boldsymbol{\Sigma}^{(1)},\boldsymbol{\Sigma}^{(2)}) = \hat{A}^* > 0.$$

Here $H$ is the generating potential (15) depending on parameters (16).

In the linear case, the invariants $I_{(1)}$–$I_{(5)}$ for crystallographic systems are written with allowance for results described in Sec. 3. Other invariants have to be considered separately. It can only be noted at the moment that only invariants (17) are present in the isotropic case.

The case most often encountered in practice is a nonlinear dependence of the generating potential on pressure $H(p)$, whereas the dependences of $H$ on the skew-symmetric part of the stress tensor $\boldsymbol{\Sigma}^{(1)}$ and deviator $\boldsymbol{\Sigma}^{(2)}$ can be assumed to be linear:

$$H = F_1(\boldsymbol{v}^{(1)}) + H_0(c_1p^2) + I_{(2)} + I_{(6)}.$$

In this case, system (24) acquires the form

$$\rho\frac{\partial}{\partial t}\boldsymbol{v}^{(1)} + \Delta_{-}\boldsymbol{\Sigma}^{(2)} + \Delta_{+}\boldsymbol{\Sigma}^{(0)} + \Delta_0\boldsymbol{\Sigma}^{(1)} = 0,$$

$$H_{pp}\,\frac{\partial}{\partial t}\,\boldsymbol{\Sigma}^{(0)} + \Delta_-\boldsymbol{v}^{(1)} = 0,$$

$$c_3\,\frac{\partial}{\partial t}\,\boldsymbol{\Sigma}^{(1)} + \Delta_0\boldsymbol{v}^{(1)} = 0,$$

$$c_2\,\frac{\partial}{\partial t}\,\boldsymbol{\Sigma}^{(2)} + \Delta_+\boldsymbol{v}^{(1)} = 0$$

with a diagonal matrix at the derivative $\partial/\partial t$, whose characteristics can be readily calculated with allowance for its invariance with respect to rotations. This is a diagonal matrix for crystals possessing cubic symmetry as well. The convenience of calculating the characteristics in the proposed system recording makes it possible to use, for instance, Godunov's difference schemes for the numerical solution of the systems of equations considered.

**Conclusions.** Advantages of the systematization proposed in the present work are the consideration of the most general case with an asymmetric stress tensor and avoiding the assumption of low strains. In addition, owing to splitting of the matrix at the derivative $\partial/\partial t$ into blocks of smaller dimension, the recording of the system of equations proposed facilitates calculation of characteristics and, hence, construction of numerical methods for solving the differential equations considered. As a whole, such an approach is not absolutely new (at least, for the linear elasticity theory), but its description differs from that commonly used.

Using the remark made at the end of Sec. 2 of this paper, one can apply the approach proposed to a more complicated system of equations of the nonlinear elasticity theory involving dissipative processes, which can lead to relaxation of the medium parameters and to changes in the crystalline structure of the material [such processes are modeled by adding a special right side to system (1)]. It is also of interest to compare the details of the approach used in the present work with available approaches to constructing invariants for the linear case (see [9, 12]), to study the conditions of convexity of the generating potential (15), i.e., well-posedness of Eqs. (1), to study the structure of the characteristics of Eqs. (1) written in the form (24), to interpret this structure in group terms, and to establish a relation between the construction of this structure and the construction usually presented in the crystal theory [13, 14].

The approach proposed can be useful for studying the equations of the nonlinear elasticity theory and for solving particular physical problems.

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## REFERENCES

1. S. K. Godunov and E. I. Romensky, *Elements of Continuum Mechanics and Conservation Laws*, Kluwer Acad. Publ., Dordrecht (2003).
2. S. K. Godunov, "The equations of elasticity with dissipation as a nontrivial example of thermodynamically consistent hyperbolic equations," *J. Hyperbol. Differ. Eq.*, **1**, No. 2, 235–249 (2004).
3. S. P. Kiselev, *Mechanics of Continuous Media* [in Russian], Novosibirsk State Technical University, Novosibirsk (1997).
4. G. Leibfried and N. Brauer, *Point Defects in Metals. Introduction to the Theory*, Springer, Heidelberg (1978).
5. S. K. Godunov and T. Yu. Mikhailova, *Presentations of Rotation Groups and Spherical Functions* [in Russian], Nauchnaya Kniga, Novosibirsk (1998).
6. S. K. Godunov and V. M. Gordienko, "Clebsch–Gordan coefficients for various choices of the bases of unitary and orthogonal presentations of the $SU(2)$ and $SO(3)$ groups," *Sib. Mat. Zh.*, **45**, No. 3, 540–557 (2004).
7. S. V. Selivanova, "Elements of the theory of presentations of the group of rotations and their application in the elasticity theory," MA Dissertation, Novosibirsk (2007).
8. S. K. Godunov and V. M. Gordienko, "Complicated structures of Galilean-invariant conservation laws," *J. Appl. Mech. Tech. Phys.*, **43**, No. 2, 175–189 (2002).
9. N. I. Ostrosablin, "On the structure of the elastic tensor and the classification of anisotropic materials," *J. Appl. Mech. Tech. Phys.*, **27**, No. 4, 600–607 (1986).

10. R. M. Garipov, "Hooke's law for single crystals," Preprint, Inst. Hydrodynamics, Sib. Div., Russ. Acad. of Sci., Novosibirsk (2005).
11. B. I. Delone, N. Padurov, and A. D. Aleksandrov, *Mathematical Fundamentals of the Structural Analysis of Crystals* [in Russian], Gostekhteorizdat, Moscow (1934).
12. N. I. Ostrosablin, "On invariants of the fourth-rank tensor of elasticity moduli," *Sib. Zh. Indust. Mat.*, **1**, No. 1, 155–163 (1998).
13. L. D. Landau and E. M. Lifshits, *Theory of Elasticity, Pergamon Press*, Oxford–New York (1970).
14. F. I. Fedorov, *Theory of Elastic Waves in Crystals* [in Russian], Nauka, Moscow (1965).